A Zero-sum Problem from Factorization Theory

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Introduction

Area of research: additive combinatorics, multiplicative ideal theory, commutative algebra, factorization theory

We study a zero-sum problem dealing with minimal zero-sum sequences of maximal length over finite abelian groups. A positive answer to this problem yields a structural description of sets of lengths with maximal elasticity in transfer Krull monoids over finite abelian groups.

Monoid of Zero-sum Sequences

Monoids of zero-sum sequences are **discrete**, **combinatorial** objects that have been investigated using methods from **additive and combinatorial number theory**. Zero-sum problems occur naturally in various branches of combinatorics, number theory, graph theory, Ramsey theory and geometry.

Let $(G,+)$ be an abelian group and $G_0 \subset G$ a subset. Let $(\mathcal{F}(G_0),\cdot)$ be the free abelian monoid with basis G_0.

- $S = g_1 \dots g_l \in \mathcal{F}(G_0)$ is called a **sequence**,
- $\sigma(S) = g_1 + \dots + g_l \in G$ is its **sum**,
- $|S| = l$ is its **length**.

Definition. The submonoid

$$\mathcal{B}(G_0) = \{\, S \in \mathcal{F}(G_0) \mid \sigma(S) = 0_G \,\} \subset \mathcal{F}(G_0)$$

is the **monoid of zero-sum sequences** over G_0. The atoms of $\mathcal{B}(G_0)$ are the **minimal zero-sum sequences**.

Davenport Constant

Erdös, Baayen and Davenport (1967–69) posed the problem to find the smallest integer l such that every sequence S over G of length $\geq l$ has a non-empty subsequence with sum zero. In the subsequent literature, it has been called $D(G)$ (**the Davenport constant**).

Definition. $\mathsf{D}(G)$ is the maximal length of a minimal zero-sum sequence over G.

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ where $r = \mathsf{r}(G)$ is **the rank** and $n_r = \exp(G)$ is **the exponent** of G. Then

- $1 + \sum_{i=1}^{r}(n_i - 1) \leq \mathsf{D}(G) \quad (\leq |G|)$
- this is an equality for p-groups and for groups with $\mathsf{r}(G) \leq 2$ (**known since 1960s**) and for some sparse series of groups only, but not known in general in terms of other group invariants.
- even less is known about the associated inverse problem (**the typical associated inverse zero-sum problem studies the structure of extremal sequences which possess no such zero-sum subsequences**).

Example.
(Due to H. Davenport) If R is the ring of integers of some algebraic number field with ideal class group (isomorphic to) G, then $D(G)$ is the maximal number of prime ideals (counted with or without multiplicity) which occur in the prime ideal decomposition of aR for each irreducible element $a \in R$.

Open problems. (on $\mathsf{D}(G)$)

- $\mathsf{D}(C_n^r) = ?$
- $\mathsf{D}(G) = ?$ when $\mathsf{r}(G) = 3$
- $\mathsf{D}(G) = ?$ when $G \cong C_p^r \oplus C_q^s$. Note that $1 + \sum(n_i - 1) \lneq \mathsf{D}(G)$ is known in this case.

Arithmetical Invariants

Arithmetical invariants measure the extent of non-uniqueness of factorizations and characterize the features that occur.

Let H be an atomic cancellative monoid, $a \in H \setminus H^{\times}$.

- The **set of lengths** of a is

$$\mathsf{L}(a) = \{\, k \in \mathbb{N} \mid a = u_1 \cdots u_k \text{ with atoms } u_i \,\}.$$

The **system of sets of lengths** is

$$\mathcal{L}(H) = \{\, \mathsf{L}(a) \mid a \in H \,\}.$$

- The **elasticity** is

$$\rho(a) = \frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} \quad \text{and} \quad \rho(H) = \sup_{a \in H} \rho(a).$$

- If $\mathsf{L}(a) = \{\, l_1 < l_2 < \cdots \}$, then the **set of distances** of a is $\Delta(a) = \{l_i - l_{i-1} \mid i\}$ and $\Delta(H) = \bigcup_{a \in H} \Delta(a)$.

Example. Let $\mathsf{C}_3 = \langle g \rangle$. Then g^3, $(-g)^3$, $g(-g)$ are minimal zero-sum sequences, i.e., elements of $\mathcal{B}(C_3)$, and

$$S = g^3(-g)^3 = (g(-g))^3$$

shows $\mathsf{L}(S) = \{2, 3\}$.

Transfer Krull Monoids

Transfer homomorphisms are constructed from a class of monoids under consideration to the one that is easier to understand. The crucial property of a transfer homomorphism is that it preserves the system of sets of lengths.

Definition. $\theta \colon H \to T$ is a **transfer homomorphism** if

(1) $T = T^{\times}\theta(H)$.

(2) For $a \in H$, $s, t \in T$, $\theta(a) = st$ implies $a = bc$ with b, $c \in H$ such that $\theta(b) = s\varepsilon^{-1}$ and $\theta(c) = \varepsilon t$ with $\varepsilon \in T^{\times}$.

How it started? (Narkiewicz 1979; Geroldinger 1988; Halter-Koch 1997) Let H be a commutative Krull monoid, G its divisor class group, and $G_0 \subset G$ the set of classes containing prime divisors. Then there exists a transfer homomorphism $\theta \colon H \to \mathcal{B}(G_0)$ into the monoid of zero-sum sequences over G_0.

Definition. A cancellative monoid H is called **transfer Krull** if there exists a transfer homomorphism $H \to \mathcal{B}(G_0)$ for some abelian group G and $G_0 \subset G$ a subset.

Examples.

(1) If R is a Dedekind domain or a Krull domain, then $H = R^{\bullet}$ is a Krull monoid. (Note that $\mathcal{C}(R)$, the usual class group, plays the role of G as in above definition)

(2) (Baeth – Smertnig 2021) Let R be a Bass ring and let $\mathsf{T}(R)$ be the monoid of isomorphism classes of torsion-free finitely generated R-modules, together with the operation induced by the direct sum. Then $\mathsf{T}(R)$ is a reduced transfer Krull monoid.
Main examples of transfer Krull monoids stem from non-commutative algebra, we mention one here as well.

(3) (Smertnig 2013) Let $\mathcal{O}_{\mathsf{K}}$ be the ring of integers in an algebraic number field K, A a central simple K-algebra, and R a classical maximal $\mathcal{O}_K$-order of A. Then $R^{\bullet}$ is transfer Krull if and only if every stably free left R-ideal is free, and if this holds then there is a transfer homomorphism $\theta \colon R^{\bullet} \to \mathcal{B}(G)$ for some finite abelian group G.

Conjecture

Conjecture. (*Geroldinger – Zhong 2018*) Every finite abelian group G, except from cyclic and elementary 2-groups, has the following property:
for every minimal zero-sum sequence $U = g_1 \cdots g_l$ over G of maximal length there are $k \in \mathbb{N}$ and minimal zero-sum sequences $U_1, \dots, U_k, V_1, \dots, V_{k+1}$ with terms from $\{g_1, \dots, g_l, -g_1, \dots, -g_l\}$ such that

$$U_1 \cdots U_k = V_1 \cdots V_{k+1}$$

Motivation

Theorem. Let H be a transfer Krull monoid over a finite (not cyclic, not elementary 2-group) abelian group G (e.g., a ring of integers with class group G). If **above conjecture holds**, then there is an $M \in \mathbb{N}$ such that every set of lengths L with maximal elasticity has the form:

$$L = L' \cup \{y, y+1, \dots, y+l\} \cup L''$$

where $y \in \mathbb{Z}$, $l \in \mathbb{N}$, $L' \subset [-M+y, -1+y]$ and $L'' \subset [y+l+1, y+l+M]$.

This result demonstrate the significance of the above conjecture that, if it holds, then all sets of lengths L with maximal elasticity $\rho(L) = \rho(H)$ are intervals apart from their globally bounded beginning (L') and end (L'') parts.

Some Known Results

The Conjecture neither holds for cyclic groups and nor for elementary 2-groups with Davenport constant greater than or equal to four.

Theorem. (*Geroldinger – Zhong 2018*) Conjecture holds for groups G with

- $\mathsf{r}(G) = 2$.
- $G \cong C_2 \oplus C_2 \oplus C_{2n}$

The proof uses the value of $\mathsf{D}(G)$ and the structure of minimal zero-sum sequences.

Theorem. (*Geroldinger – Zhong 2018*) Conjecture holds for groups isomorphic to $G \cong C_{p^k}^r$, $p^k > 2$.

The proof uses the value of $\mathsf{D}(G)$, but the structure of minimal zero-sum sequences is not needed.

Our Main Result

Theorem. (*B.– Geroldinger – Zhong 2020*) Conjecture holds for the following non-cyclic finite abelian groups G.

(a) G is a p-group such that $\gcd(\exp(G) - 2, \mathsf{D}(G) - 2) = 1$.

(b) $G \cong C_{p^{s_1}}^{r_1} \oplus C_{p^{s_2}}^{r_2}$, where p is a prime and $r_1, r_2, s_1, s_2 \in \mathbb{N}$ such that s_1 divides s_2.

(c) G is a group with $\exp(G) = pq$, where p, q are distinct primes satisfying one of the following four properties:

(i) $\gcd(pq - 2, \mathsf{D}(G) - 2) = 1$.

(ii) $\gcd(pq - 2, p + q - 3) = 1$.

(iii) $q = 2$ and $p - 1$ is a power of 2.

(iv) $q = 2$ and $\mathsf{r}_p(G) = 1$.

(d) G is a group with $\exp(G) \in [3, 11] \setminus \{8\}$.

Note that $\mathsf{D}(G)$ and the structure of minimal zero-sum sequences of maximal length (namely $\mathsf{D}(G)$) is not known in most of the cases discussed in the above theorem.

References

(1) A. Bashir and A. Geroldinger and Q. Zhong, *On a zero-sum problem arising from factorization theory*, to appear in Combinatorial and Additive Number Theory IV, Springer 2021.

(2) A. Geroldinger and Q. Zhong, *Long sets of lengths with maximal elasticity*, Can. J. Math. **70** (2018), 1284 – 1318.

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